9.2 (continued)

last time: $f(t)$ period $2L$ given on $-L<t<L$

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right]$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

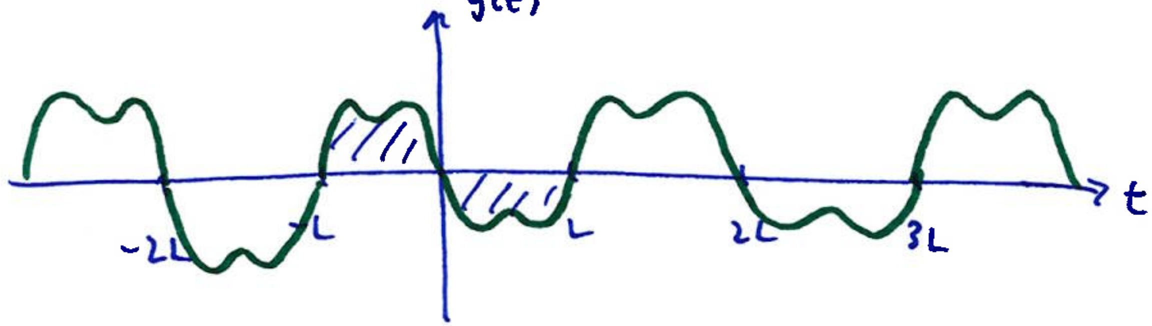
$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

usually, we prefer $f(t)$ period $2L$ but given on $0<t<2L$

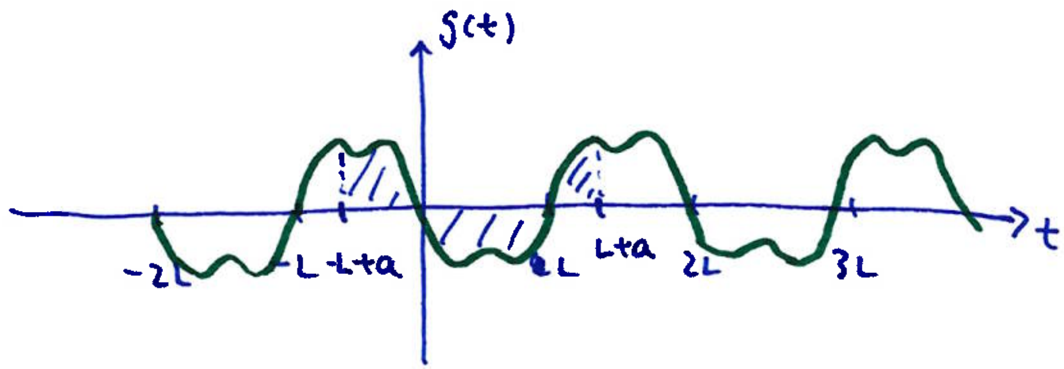
first formula stays the same — no assumption on the interval of t

the other two may be affected by the change of integration interval

$g(t)$ period $2L$, it looks like



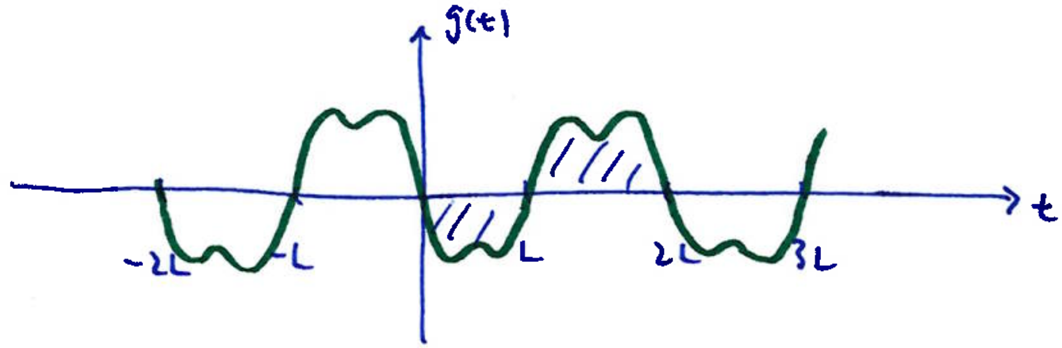
$\int_{-L}^{L} g(t)\,dt$ is the area of the ~~shifted~~ shaded region.



$\int_{-L+a}^{L+a} g(t)\,dt$ is the area of the shaded region

notice they are the same.

if we make $a = L$ then we get



$$\int_0^{2L} g(t)\,dt = \int_{-L}^{L} g(t)\,dt = \int_{-L+a}^{L+a} g(t)\,dt$$

(more natural interval)

the same as long as the integration interval is 2L (one period)

→ g(t) must be periodic w/ period 2L

look at $a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(t)\cos\left(\frac{n\pi t}{L}\right)}_{\text{collectively must have 2L as period}} dt$

f(t) has period 2L by definition

$\cos\left(\frac{n\pi t}{L}\right)$ has period of $\frac{2\pi}{n\pi/L} = \frac{2L}{n}$ fundamental period

→ but any integer multiple is also a period

→ so 2L is a period of $\cos\left(\frac{n\pi t}{L}\right)$

so, $f(t)\cos\left(\frac{n\pi t}{L}\right)$ also has period $2L$

therefore, $a_n = \frac{1}{L}\int_0^{2L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$

$$b_n = \frac{1}{L}\int_0^{2L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

alternatively, we can choose to work with period $P = 2L$

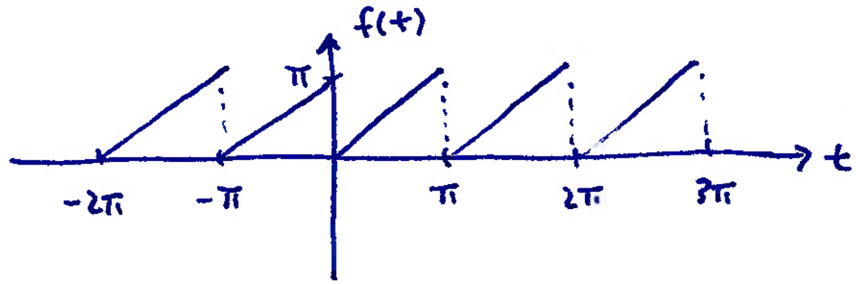
change $L = P/2$

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2n\pi}{P}t\right) + b_n\sin\left(\frac{2n\pi}{P}t\right)\right]$$

$$a_n = \frac{2}{P}\int_0^{P} f(t)\cos\left(\frac{2n\pi}{P}t\right)dt$$

$$b_n = \frac{2}{P}\int_0^{P} f(t)\sin\left(\frac{2n\pi}{P}t\right)dt$$

example $\quad f(t) = t \quad 0 < t < \pi \quad$ period π $\quad (L = \pi/2)$



$$a_0 = \frac{1}{\pi/2} \underbrace{\int_0^\pi t\,dt}_{\text{area}} = \frac{2}{\pi}\left(\frac{1}{2}\cdot\pi\cdot\pi\right) = \pi$$

$$a_n = \frac{1}{\pi/2} \int_0^\pi t\cos(2nt)\,dt = \dots = 0$$

$$b_n = \frac{1}{\pi/2} \int_0^\pi t\sin(2nt)\,dt = \dots = -\frac{1}{n}$$

$a_0/2$

$$t \sim \frac{\pi}{2} + \sum_{n=1}^{\infty} -\frac{1}{n}\sin(2nt)$$

$$t \sim \frac{\pi}{2} - \sin(2t) - \frac{1}{2}\sin(4t) - \frac{1}{3}\sin(6t) - \frac{1}{4}\sin(8t) - \dots$$

let's evaluate a $t = \frac{\pi}{4}$

Since $t = \frac{\pi}{4}$ is where $f(t)$ is continuous, the Fourier series converges to the value of $f(t)$

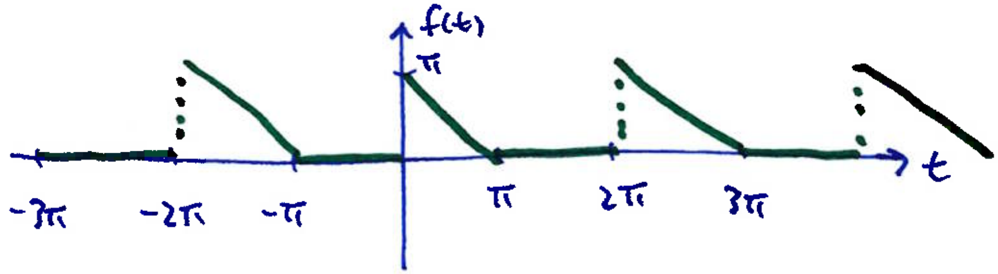
$$\frac{\pi}{4} = \frac{\pi}{2} - 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$$

$$-\frac{\pi}{4} = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$$

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots}$$ Leibniz series

<u>example</u>

$$f(t) = \begin{cases} \pi - t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$$ period 2π $(L = \pi)$



$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t)\,dt = \frac{1}{\pi}\left(\frac{1}{2}\cdot\pi\cdot\pi\right) = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\,dt = \cdots = \frac{1-(-1)^n}{n^2\pi} = \begin{cases} 0 & n \text{ is even} \\ \frac{2}{n^2\pi} & n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\,dt = \cdots = \frac{1}{n}$$

$$f(t) \sim \frac{\pi}{4} + \sum_{n \text{ odd}}^{\infty} \frac{2}{n^2\pi}\cos(nt) + \sum_{n=1}^{\infty} \frac{1}{n}\sin(nt)$$

$t=0$

at $t=0$, function is NOT continuous

Fourier series converges to $\dfrac{f(0^-)+f(0^+)}{2} = \dfrac{\pi}{2}$

$$\frac{\pi}{2} = \frac{\pi}{4} + \sum_{n \text{ odd}}^{\infty} \frac{2}{n^2\pi}$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots\right)$$

simplify

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$$ sum of reciprocals of odd squares

on HW, you are asked to show $\dfrac{\pi^2}{6} = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \dfrac{1}{5^2} + \dfrac{1}{6^2} + \cdots$

(Basel problem)